

# Clocks' synchronization without round-trip conditions

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Poincaré-Einstein's synchronization convention is transitive, and thus leads to a consistent synchronization, only if some form of round-trip property is satisfied. An improved version is given here which does not suffer from this limitation and which therefore may find application in physics, computer science and communications theory. As for the application to physics, the round-trip condition required by the Poincaré-Einstein's synchronization convention corresponds to a vanishing Sagnac effect and thus to the selection of an irrotational frame. The corrected method applies also to rotating frames and shows that there is a consistent synchronization for every given measure on space. The correction to Poincaré-Einstein's amounts to an average of the Sagnac holonomy over all the possible triangular paths. The mathematics used is reminiscent of Alexander cohomology theory.

## 1 Introduction

In the middle of the XIX century the telegraphic technology began to flourish. Cables were laid across the oceans and the possibility of communicating Greenwich's time to Americas allowed unprecedented longitude measurements [1]. In order to increase the precision the engineers took into account the one-way transmission time. This time was set as half the two-way time arguing that the signal moves at the same velocity independently of the direction taken along the cable.

Conceptually, measurements of one-way velocity make sense only after a suitable synchronization of distant clocks, thus we might more properly say that the engineers were using a synchronization method that made the speed of the signal on the cable isotropic.

In 1904 Poincaré [2, 3] and in 1905 Einstein [4] extended the method to light signals, so that it is now generally known as Einstein's (1905) or Poincaré-Einstein's synchronization method (convention) (for an account of the different

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synchronization methods introduced by Einstein see [5], they all coincide if property  $z = \mathbf{0}$  below holds). In short the method allows to find that time coordinate that makes the one-way-velocity of light isotropic. Of course such time coordinate need not exist, a well known fact that is at the origin of the Sagnac effect in rotating frames [6, 7, 8, 9, 10, 11, 12]. One should therefore impose some condition that allows the consistent application of Einstein's convention. This condition is usually a round-trip property which physically demands that the frame be irrotational. We shall return to these conditions in the next sections.

Einstein's synchronization procedure answers to the practical need of a time coordinatization of spacetime. Many methods can be conceived in that respect but none is so general that it can be applied in any circumstance in which the problem of spreading time over space makes sense. For instance, Einstein's method is really effective only in the inertial frames of special relativity or in extended frames that can be approximated by those. In this work, a generalization of Poincaré-Einstein method is given which widens its applicability to rotating frames in curved spacetime provided one restricts to suitable surfaces with vanishing relative redshift. Such a generalization is important not only for a deeper theoretical understanding of the synchronization process but also because the planet in which we live, the earth, is a rotating frame.

Other fields of application are computer science and communication theory. Extended computational networks need to be synchronized and the synchronization method that is universally adopted is that of Poincaré-Einstein [13, 14]. Unfortunately, these systems may violate the round-trip condition that a consistent application of this convention requires. In this respect, the modified convention proposed in this work can prove particularly useful. Note in particular that in all this work the nature of the signal is not specified, it can be light propagating in vacuum, sound propagating in the air, or it can be an electric signal propagating along copper wires.

When dealing with a spacetime manifold the metric signature is  $(- + ++)$ .

## 2 The abstract framework

Let us introduce a mathematical framework which will allow us to deal with the problem of synchronization without the need of making reference to a previously existing theory. It will prove particularly general so that special and general relativity will be considered as special cases. At first the mathematical framework may seem somewhat abstract but the price paid in abstractness makes the exposition of the arguments shorter as it saves repetitions of sentences like "consider a signal starting from ... arriving at ... reflected back ...".

**Definition 2.1.** A *synchronization structure*  $(M, T, \pi, \pi_{TS}, P, p, S)$  is given by: a set  $S$  called *the space*, each element  $s \in S$  being called a *space point* or *clock*. A *spacetime*  $M$ , whose elements are called *events*, defined as the disjoint union  $M = \bigcup_{s \in S} \mathbb{E}_s$  where  $\mathbb{E}_s$  are one-dimensional affine spaces over one-dimensional vector spaces  $T_s$ , that is given two elements  $e_1, e_2 \in \mathbb{E}_s$  the difference makes sense and belongs to  $T_s$ . The difference is called *time interval* of the events  $e_1$

and  $e_2$  *happening* at  $s$ . The time interval is not a real number because a unit of measure of time must first be defined at  $s \in S$ . The *unit of measure* is a particular time interval, i.e. an element  $\tau_s \in T_s$  chosen at  $s$ . If this privileged element is given, the measured time interval is the number  $t_{12} \in \mathbb{R}$  such that  $e_2 - e_1 = t_{12}\tau_s$ . The space of units of measure is  $T = \bigcup_{s \in S} T_s$ , and  $\pi_{TS} : T \rightarrow S$  is the canonical projection. A unit of measure is chosen at each space point if a section  $\tau : S \rightarrow T$ ,  $s \rightarrow \tau_s$ , is given. Moreover,  $T$  is *time oriented* in the sense that a choice of *positive* halve for  $T_s$  has been made at each  $s \in S$  (which makes the inequality  $e' - e \geq 0$  meaningful if  $e$  and  $e'$  belong to the same fiber).

Next, there is a natural projection  $\pi : M \rightarrow S$  which assigns to  $e \in M$ , the point  $s$  such that  $e \in \mathbb{E}_s$ . There is also the *propagation* map  $P : M \times S \rightarrow M$  such that, denoting with  $\pi_M$  and  $\pi_S$  the projections of  $M \times S$  on  $M$  and  $S$  respectively,  $\pi \circ P = \pi_S$ . In short, given the event  $e_{s_1} \in M$ ,  $\pi(e_{s_1}) = s_1$ , and  $s_2 \in S$ , the map sends the pair  $(e_{s_1}, s_2)$  to a new event  $e_{s_2} = P(e_{s_1}, s_2)$  which projects on  $s_2$ . In the same way, there is the propagation map  $p : T \times S \rightarrow T$ , which for any given interval  $\tau_{s_1} \in T_{s_1}$ , and point  $s_2 \in S$  gives an interval  $p(\tau_{s_1}, s_2) \in T_{s_2}$ .

Defined for every  $k \in \mathbb{N}$  the maps

$$\begin{aligned} P^k : M \times \underbrace{S \times \cdots \times S}_{k \text{ factors}} &\rightarrow M \\ p^k : T \times \underbrace{S \times \cdots \times S}_{k \text{ factors}} &\rightarrow T \end{aligned}$$

as follows

$$\begin{aligned} P^k(e_{s_0}, s_1, s_2, \dots, s_k) &= P(P(\dots P(P(e_{s_0}, s_1), s_2), \dots, s_{k-1}), s_k) & \text{if } k > 0 \\ P^0(e) &= e & \text{if } k = 0 \end{aligned}$$

and analogously for  $p$ , on  $P$  are imposed the conditions

- (a) (Fermat) Given a sequence of points  $s_0, s_1$  and  $s_2$ ,  $P$  satisfies

$$P^2(e_{s_0}, s_1, s_2) - P(e_{s_0}, s_2) \geq 0. \quad (1)$$

- (b) (Causality) Given a cyclic sequence of points  $s_0, s_1, \dots, s_k = s_0$ ,  $k \geq 1$ ,  $P$  satisfies

$$P^k(e_{s_0}, s_1, s_2, \dots, s_k) - e_{s_0} \geq 0, \quad (2)$$

where the equality holds iff  $s_0 = s_1 = \dots = s_{k-1}$ , in particular  $P(e_s, s) = e_s$ .

- (c) ( $\mathbf{z} = \mathbf{0}$ ) The map  $P$  is an affine map, that is for every  $e_{s_1} \in \mathbb{E}_{s_1}$   $s_2 \in S$ , and  $\tau_{s_1} \in T_{s_1}$  it is

$$P(e_{s_1} + \tau_{s_1}, s_2) = P(e_{s_1}, s_2) + p(\tau_{s_1}, s_2). \quad (3)$$

Stated in another way, if  $e_{s_1}, e'_{s_1} \in \mathbb{E}_{s_1}$  and  $s_2 \in S$ , then

$$P(e'_{s_1}, s_2) - P(e_{s_1}, s_2) = p(e'_{s_1} - e_{s_1}, s_2),$$

and  $p$  is an injective linear map which preserves the time orientation of  $T$ , that is for every  $\tau_{s_1} \in T_{s_1}$ ,  $s_2 \in S$ , and  $\alpha \in \mathbb{R}$ ,  $p(\tau_{s_1}, s_2)$  is positive iff  $\tau_{s_1}$  is positive and

$$p(\alpha\tau_{s_1}, s_2) = \alpha p(\tau_{s_1}, s_2).$$

(d) (no self redshift) Given a cyclic sequence of points  $s_0, s_1, \dots, s_k = s_0$ ,  $k \geq 1$ ,  $p$  satisfies

$$p^k(\tau_{s_0}, s_1, s_2, \dots, s_k) - \tau_{s_0} = 0. \quad (4)$$

A short definition can be provided as follows

**Definition 2.2.** A synchronization structure is an affine bundle  $\pi : M \rightarrow S$  associated to a vector bundle  $\pi_{TS} : T \rightarrow S$  with one dimensional fibers, and an affine map  $P : M \times S \rightarrow M$ , associated to a linear map  $p : T \times S \rightarrow T$ , which satisfies conditions (a), (b), (c) and (d) above.

A simple consequence of (c) is that  $P^k$  is an affine map and  $p^k$  is a linear map.

Light never enters explicitly the theory so that it does not play any privileged role (indeed,  $S$  need not even be a manifold). Depending on the context other signals propagating on space but of different nature could be considered. The very interpretation of  $P$  as coming from the propagation of a signal is not needed for the development of the theory but will be often cited in order to fix the ideas. Thus, in the most straightforward interpretation,  $P(e_{s_1}, s_2)$  represents the event of arrival at  $s_2$  of a light beam sent at event  $e_{s_1}$  towards  $s_2$ . The fact that  $P(e_s, s) = e_s$  means that if  $s_1 = s_2$ , then the event of departure coincides with that of arrival.

The Fermat's condition (a) is not really restrictive, indeed in most applications one would have a signal propagating on a suitable space  $S$ , then the propagation map  $P$  would be obtained imposing condition (a). That is, given  $e_{s_1}$  and  $s_2$  one identifies  $e_{s_2} = P(e_{s_1}, s_2)$  with the first event (or the upper lower bound) on  $\mathbb{E}_s$  (in its natural order) which can be influenced from  $e_{s_1}$ . This definition makes (a) automatically satisfied. Note also that the signal may follow different 'paths' all reaching the same event on  $\mathbb{E}_s$ , thus this procedure selects an arrival event, not a 'path' over which the signal propagates. The concept of 'path' for the propagating signal may make no sense in the physical model to which the synchronization structure applies. For instance, in general relativity, in the optical geometric limit, it makes sense to speak of the path of a light beam, otherwise the concept of light beam and path do not make sense, although the synchronization structure remains meaningful.

The inequality (2) expresses a causality requirement: if the signal covers a closed path then it returns at an event which comes after the departure on  $\mathbb{E}_{s_1}$ .

Note that the time difference makes sense only if the events belong to the same fiber  $\mathbb{E}_s$ . The time interval between events that do not happen at the same point is not defined. The basic problem of synchronization theory is the

*synchronization problem* namely the problem of finding a general but natural method for foliating  $M$  into (simultaneity) slices, a slice being a section  $\sigma : S \rightarrow M$  of the bundle  $\pi : M \rightarrow S$ . Often this problem is considered only after a suitable solution to the *syntonization* problem has been found. The *syntonization* problem asks to determine a natural method for selecting a section  $\tau : S \rightarrow T$  of the bundle  $\pi_{TS} : T \rightarrow S$ .

We now seek a solution to the syntonization problem which makes use only of the already introduced synchronization structure.

The syntonization problem can be solved by choosing a time unit  $\tau_{s_0}$  at  $s_0$  and defining the time unit at  $\tilde{s}$  as that obtained by the finite repeated application of  $p$  over a polygonal path with endpoints  $s_0$  and  $\tilde{s}$  (in practice two signals separated by a time interval  $\tau_{s_0}$  are sent from  $s_0$  along the polygonal path, and the time interval given by the arrival events at  $\tilde{s}$  gives the unit at  $\tilde{s}$ ). This method in order to be meaningful must be independent of the polygonal path which connects  $s_0$  to  $\tilde{s}$ . This fact is guaranteed by (d). Indeed, if there were two polygonal paths  $\gamma_1$  and  $\gamma_2$  to  $\tilde{s}$  that would bring  $\tau_{s_0}$  to two vectors  $\tau_{\tilde{s}}^1$  and  $\tau_{\tilde{s}}^2$ , then by applying  $p$  recursively along  $\gamma_1^{-1}$  we would get, using (d) for  $\gamma_1^{-1} \circ \gamma_1$  and  $\gamma_1^{-1} \circ \gamma_2$  the same vector  $\tau_{s_0}$ , which by the injectivity of  $p$  implies  $\tau_{\tilde{s}}^1 = \tau_{\tilde{s}}^2$ . It can also be easily checked that the choice of  $s_0$  is irrelevant and that there remains only an arbitrariness in the choice of  $\tau_{s_0}$ . This overall arbitrary scale factor independent of the location is natural in the choice of a unit of measure.

The just constructed section  $\tau : S \rightarrow T$ , shares the property, for every  $s_1, s_2 \in S$

$$p(\tau_{s_1}, s_2) = \tau_{s_2}, \quad (5)$$

and provides a solution to the syntonization problem. Of course this solution has been possible thanks to condition (d). One could generalize the synchronization structure by dropping condition (d). This would lead to a fairly more general theory in which both the syntonization and the synchronization problems would become non-trivial. In this work, we shall keep condition (d) on the ground of simplicity and also because it will be sufficient for the proposed applications.

As a consequence, throughout this work we shall omit reference to the application  $p$  assuming that a section  $\tau$  with property (5) has been chosen. Thus time intervals can be identified with real numbers, and equations such as (3) can be written more sloppily

$$P(e_{s_1} + \tau_{s_1}, s_2) = P(e_{s_1}, s_2) + \tau_{s_1}. \quad (6)$$

The reader interested in syntonization issues in general relativity may also consult [15, 16].

*Remark 2.3.* The spacetime of general relativity, and hence of special relativity, fits into this setting once a congruence of timelike worldlines is defined (the frame). The space of the worldlines of the congruence plays the role of  $S$ , the congruence defining a notion of “rest” with respect to the frame. At each point  $s$  of the frame a clock at rest, i.e. whose worldline coincides with  $s$ , measures a proper time which is defined only up to an additive constant (resynchronization).

However, for any pair of events on the same worldline the proper time interval between the events makes sense, which provides each worldline with an affine structure.

In general relativity given the timelike congruence the map  $P$  is defined through the Fermat's principle [17, 18], and follows from the existence of the light cone structure on  $M$ . It suffices to define  $P(e_{s_1}, s_2)$  as the intersection of the light cone issuing from  $e_{s_1}$  with the worldline  $\pi^{-1}(s_2)$  of  $s_2$ , with the rule that if it has more than one event then the one with the smallest value of  $s_2$ 's proper time must be taken. If there is no intersection then the two worldlines are separated by a particle horizon. In this case the frame given by the congruence is too general to be included in the above framework. Nevertheless, at least locally the timelike congruence leads to a synchronization structure.

A natural foliation does not seem to exist in general. Vorticity free congruences are an exception as they are hypersurface orthogonal. This kind of orthogonal foliation, whenever it exists, is obtained by the local application of the Einstein synchronization convention [11]. The absence of vorticity corresponds to the absence of a Sagnac effect. For more details see [7, 8, 11].

The condition (c), also denoted  $\mathbf{z} = \mathbf{0}$  for reason that will be clear in a moment, is physically and mathematically demanding but it has a simple justification. In the light propagation interpretation it states that two light beams sent from  $s_1$ , the second after  $\Delta t$  from the departure of the first, reach  $s_2$  at times separated by the same interval as measured by  $s_2$ . Considering that the electromagnetic phase is constant over the light beam, i.e. the number of maximums on the monochromatic wave is the same for the observers placed at  $s_1$  or  $s_2$ , this condition means that there is no redshift between the two points, hence the notation  $\mathbf{z} = \mathbf{0}$ . Another legitimate point of view regards  $\mathbf{z} = \mathbf{0}$  as a condition of time homogeneity, or translational time invariance as it is suggested by Eq. (3).

The condition  $\mathbf{z} = \mathbf{0}$  is not fulfilled by all the timelike congruences over a spacetime. However, assume that the congruence is generated by a nowhere vanishing timelike conformal Killing field  $k$

$$L_k g_{\alpha\beta} = \frac{\partial_k(k \cdot k)}{k \cdot k} g_{\alpha\beta}.$$

Defined  $\hat{g} = g/(-k \cdot k)$  since  $L_k k = 0$  it is easy to check  $L_k \hat{g} = 0$ , thus  $k$  is a normalized (as  $\hat{g}(k, k) = -1$ ) Killing vector for the spacetime  $(M, \hat{g})$ .

It is now easy to check that  $\mathbf{z} = \mathbf{0}$  is satisfied on  $(M, \hat{g})$  for the frame generated by  $k$ . Indeed, the propagation of light on  $(M, g)$  coincides with that of  $(M, \hat{g})$  as they have the same unparametrized lightlike geodesic. Moreover, in a stationary spacetime the redshift between event  $e_1$  and event  $e_2$  at the endpoints of a lightlike geodesic is given by the ratio  $1+z = \sqrt{\hat{g}(k, k)(e_2)/\hat{g}(k, k)(e_1)}$  which in the spacetime  $(M, \hat{g})$  gives unity as required.

Thus the problem of time coordinatization for the triple  $(M, g, k)$  where  $k$  is a conformal Killing field can be reduced to that for the triple  $(M, \hat{g}, k)$ .

One may wonder whether condition  $\mathbf{z} = \mathbf{0}$  is physically too restrictive. Indeed, this condition is restrictive but a solution of the foliation problem in this

case would already represents a considerable progress. It must be taken into account that the surface of the earth is an equipotential slice and as such there is no redshift between its points [19]. The usual “common view” GPS method of synchronization [20, 21, 10, 22] does not provide the general and natural method of synchronization seeked in this work. Indeed, it depends on many details of the earth geoid, on the spacetime metric, on the satellites orbits and so on. It provides an efficient but ad hoc solution, which requires a lot of information which does not enter into the statement of the problem as expressed by the synchronization structure. Indeed, as we shall see, a different and more appealing solution exists which only makes use of the already introduced mathematical structure. In this sense the new solution is far more general and natural. Moreover, as we have already pointed out, the spacetimes admitting a conformal Killing field can be reduced to the case  $z = \mathbf{0}$ , so that many cosmological applications will be included too.

*Remark 2.4.* Apart from general and special relativity there is another related example which can be recasted in the introduced mathematical framework and which is of primary importance for the physical interpretation of the theory. Let the set  $S$  be the finite set of clocks of computers disseminated on the surface of the earth and connected among themselves through the internet. The same mathematical framework can describe a smaller LAN, for instance made of few but very stable reference clocks connected through intercontinental optical fibers. As a matter of fact some of these servers may be connected with optical fibers, others with ordinary cables, other with electromagnetic signal propagating in the atmosphere. The theory is very versatile and works also in these cases. The only possible problem is that signals propagating in the atmosphere would depend on the pressure, temperature and humidity of the air. Since they are time dependent the additional stability property  $z = \mathbf{0}$  would not be satisfied.

In this web based application *the time* it takes an information packet to move from one internet node to the next may depend considerably not only on the distance between the nodes but also on the nature of the wires and on the speed of the computer servers at the nodes. The nice fact is that the theory developed here is completely independent of these details. Notice that the concept of time mentioned in the sentence above and italicized is a kind of external time which has nothing to do with the time of the clocks at the nodes prior to synchronization. The very fact that the cables connecting two nodes are, say, *slow* makes almost no sense in the theory, because tacitly assumes a prior synchronization of the clocks i.e. a “time” above the one that we wish to construct. Of course it may make sense to speak of such a time, given a wider theory, but not from the point of view of the theory that we are developing. The theory might not apply if  $z = \mathbf{0}$  is broken in some way, for instance this can happen if the reply of the servers depends on the chaotic traffic passing through them, but in general the *slow* nature of the signal propagation is irrelevant.

### 3 The functions $r$ and $w$ .

Consider the function  $r : S \times S \rightarrow [0, +\infty)$  defined by

$$r(s_0, s_1) = P^2(e_{s_0}, s_1, s_0) - e_{s_0}, \quad (7)$$

and the function  $w : S \times S \times S \rightarrow \mathbb{R}$  defined by

$$w(s_0, s_1, s_2) = P^3(e_{s_0}, s_1, s_2, s_0) - P^3(e_{s_0}, s_2, s_1, s_0), \quad (8)$$

the property  $\mathbf{z} = \mathbf{0}$  implies that both  $r$  and  $w$  are well defined as they do not depend on the choice of  $e_{s_0} \in \mathbb{E}_{s_0}$ . It is  $r(s_0, s_1) = 0$  iff  $s_0 = s_1$ .

*Remark 3.1.* Physically  $r(s_0, s_1)$  represents the two-way echo time. In the computer web interpretation it is the result that computer  $s_0$  obtains after “pinging”  $s_1$ . The function  $w$  can instead be interpreted, in general relativity, as the well known Sagnac effect over a “triangle” of vertices  $s_0, s_1, s_2$ . The important point is that these two functions are observable. From them it is possible to obtain a new synchronization method. Note that Einstein’s method uses only  $r$  and assumes  $w = 0$ , see section 4.

The next lemma gives a tool for simplifying some lengthy expressions

**Lemma 3.2.** *Let  $k \geq 3$  then for every  $s_1, s_2, s_3, s_4 \in S$ ,*

$$P^k(\dots, s_3, s_2, s_1, s_2, s_4, \dots) = P^{k-2}(\dots, s_3, s_2, s_4, \dots) + r(s_2, s_1)$$

*Proof.*

$$\begin{aligned} P^k(\dots, s_3, s_2, s_1, s_2, s_4, \dots) &= P^{k-i}(P^i(\dots, s_3, s_2, s_1, s_2), s_4, \dots) \\ &= P^{k-i}(P^{i-2}(\dots, s_3, s_2) + [P^i(\dots, s_3, s_2, s_1, s_2) - P^{i-2}(\dots, s_3, s_2)], s_4, \dots) \\ &= P^{k-i}(P^{i-2}(\dots, s_3, s_2), s_4, \dots) + [P^i(\dots, s_3, s_2, s_1, s_2) - P^{i-2}(\dots, s_3, s_2)] \\ &= P^{k-2}(\dots, s_3, s_2, s_4, \dots) + [P^2(P^{i-2}(\dots, s_3, s_2), s_1, s_2) - P^{i-2}(\dots, s_3, s_2)] \\ &= P^{k-2}(\dots, s_3, s_2, s_4, \dots) + r(s_2, s_1) \end{aligned}$$

□

**Theorem 3.3.** *The function  $r$  is symmetric.*

*Proof.* Recall that

$$r(s_1, s_0) = P^2(e_{s_1}, s_0, s_1) - e_{s_1}.$$

Since  $P$  preserves the affine structure

$$\begin{aligned} r(s_1, s_0) &= P(P^2(e_{s_1}, s_0, s_1), s_0) - P(e_{s_1}, s_0) \\ &= P^2(P(e_{s_1}, s_0), s_1, s_0) - P(e_{s_1}, s_0) = r(s_0, s_1). \end{aligned}$$

□

**Theorem 3.4.** *The function  $w$  is skew-symmetric.*



*Proof.* The relation  $w(s_0, s_1, s_2) = -w(s_0, s_2, s_1)$  is obvious thus it suffices to prove the cyclicity  $w(s_0, s_1, s_2) = w(s_1, s_2, s_0)$ . First note that  $w(s_1, s_2, s_0) = P^3(e_{s_1}, s_2, s_0, s_1) - P^3(e_{s_1}, s_0, s_2, s_1)$  but  $e_{s_1}$  can be chosen arbitrarily, thus take  $e_{s_1} = P(e_{s_0}, s_1)$  then

$$\begin{aligned} w(s_1, s_2, s_0) &= P^4(e_{s_0}, s_1, s_2, s_0, s_1) - P^4(e_{s_0}, s_1, s_0, s_2, s_1) \\ &= P^4(e_{s_0}, s_1, s_2, s_0, s_1) - P^2(P^2(e_{s_0}, s_1, s_0), s_2, s_1) \\ &= P^4(e_{s_0}, s_1, s_2, s_0, s_1) - P^2(e_{s_0}, s_2, s_1) - [P^2(e_{s_0}, s_1, s_0) - e_{s_0}], \end{aligned}$$

using the translational invariance of  $P$

$$\begin{aligned} w(s_0, s_1, s_2) &= P^4(e_{s_0}, s_1, s_2, s_0, s_1) - P^4(e_{s_0}, s_2, s_1, s_0, s_1) \\ &= P^4(e_{s_0}, s_1, s_2, s_0, s_1) - P^2(P^2(e_{s_0}, s_2, s_1), s_0, s_1) \\ &= P^4(e_{s_0}, s_1, s_2, s_0, s_1) - P^2(e_{s_1}, s_0, s_1) - [P^2(e_{s_0}, s_2, s_1) - e_{s_1}] \\ &= P^4(e_{s_0}, s_1, s_2, s_0, s_1) - P^3(e_{s_0}, s_1, s_0, s_1) - [P^2(e_{s_0}, s_2, s_1) - e_{s_1}] \\ &= P^4(e_{s_0}, s_1, s_2, s_0, s_1) - P(P^2(e_{s_0}, s_1, s_0), s_1) - [P^2(e_{s_0}, s_2, s_1) - e_{s_1}] \\ &= P^4(e_{s_0}, s_1, s_2, s_0, s_1) - P(e_{s_0}, s_1) - [P^2(e_{s_0}, s_1, s_0) - e_{s_0}] \\ &\quad - [P^2(e_{s_0}, s_2, s_1) - e_{s_1}] \\ &= P^4(e_{s_0}, s_1, s_2, s_0, s_1) - P^2(e_{s_0}, s_2, s_1) - [P^2(e_{s_0}, s_1, s_0) - e_{s_0}], \end{aligned}$$

thus  $w(s_0, s_1, s_2) = w(s_1, s_2, s_0)$  as claimed.  $\square$

**Theorem 3.5.** *For every choice of  $s_1, s_2, s_3, s_4 \in S$ , the function  $w$  satisfies*

$$w(s_2, s_3, s_4) - w(s_3, s_4, s_1) + w(s_4, s_1, s_2) - w(s_1, s_2, s_3) = 0. \quad (9)$$

*Remark 3.6.* In analogy with homology or Cohomology theory Eq. (9) may be called *the 2-cocycle condition*. The cochains considered here are almost equivalent to those considered by the Alexander-Kolmogorov cohomology theory [23, Sect. 6.4]. However, here a condition on the cochains is missed so that all our cohomology groups are trivial. As we shall see,  $w$  is not only a 2-cocycle but also a 2-coboundary (Eq. (21) and theorem 5.3).

*Proof.* Note that given arbitrary  $e_{s_1}, e'_{s_1} \in \mathbb{E}_{s_1}$  we can write

$$w(s_1, s_2, s_3) = [P^3(e_{s_1}, s_2, s_3, s_1) - e_{s_1}] - [P^3(e'_{s_1}, s_3, s_2, s_1) - e'_{s_1}]$$

indeed the terms in the square brackets do not depend on the choice of  $e_{s_1}$  or  $e'_{s_1}$ , and if  $e_{s_1} = e'_{s_1}$  the right-hand side reduces to Eq. (8). In particular, in this case we choose  $e'_{s_1} = P^6(e_{s_1}, s_2, s_4, s_1, s_4, s_3, s_1)$ . In the analogous equation

$$w(s_1, s_3, s_4) = [P^3(e''_{s_1}, s_3, s_4, s_1) - e''_{s_1}] - [P^3(e'''_{s_1}, s_4, s_3, s_1) - e'''_{s_1}]$$

we choose  $e''_{s_1} = P^3(e_{s_1}, s_2, s_3, s_1)$  and  $e'''_{s_1} = P^3(e_{s_1}, s_2, s_4, s_1)$ . In the equation

$$w(s_1, s_4, s_2) = [P^3(e''''_{s_1}, s_4, s_2, s_1) - e''''_{s_1}] - [P^3(e'''''_{s_1}, s_2, s_4, s_1) - e'''''_{s_1}]$$

we choose  $e_{s_1}'''' = P^6(e_{s_1}, s_2, s_3, s_1, s_3, s_4, s_1)$  and  $e_{s_1}''''' = e_{s_1}$ . Thus

$$\begin{aligned}
& w(s_1, s_2, s_3) + w(s_1, s_3, s_4) + w(s_1, s_4, s_2) = [P^3(e_{s_1}, s_2, s_3, s_1) - e_{s_1}] \\
& - [P^9(e_{s_1}, s_2, s_4, s_1, s_4, s_3, s_1, s_3, s_2, s_1) - P^6(e_{s_1}, s_2, s_4, s_1, s_4, s_3, s_1)] \\
& + [P^6(e_{s_1}, s_2, s_3, s_1, s_3, s_4, s_1) - P^3(e_{s_1}, s_2, s_3, s_1)] \\
& - [P^6(e_{s_1}, s_2, s_4, s_1, s_4, s_3, s_1) - P^3(e_{s_1}, s_2, s_4, s_1)] \\
& + [P^9(e_{s_1}, s_2, s_3, s_1, s_3, s_4, s_1, s_4, s_2, s_1) - P^6(e_{s_1}, s_2, s_3, s_1, s_3, s_4, s_1)] \\
& - [P^3(e_{s_1}, s_2, s_4, s_1) - e_{s_1}] \\
& = P^9(e_{s_1}, s_2, s_3, s_1, s_3, s_4, s_1, s_4, s_2, s_1) - P^9(e_{s_1}, s_2, s_4, s_1, s_4, s_3, s_1, s_3, s_2, s_1)
\end{aligned}$$

Define  $e_{s_2} = P(e_{s_1}, s_2)$  then

$$\begin{aligned}
& P^9(e_{s_1}, s_2, s_3, s_1, s_3, s_4, s_1, s_4, s_2, s_1) - P^9(e_{s_1}, s_2, s_4, s_1, s_4, s_3, s_1, s_3, s_2, s_1) \\
& = P^7(e_{s_2}, s_3, s_1, s_3, s_4, s_1, s_4, s_2) - P^7(e_{s_2}, s_4, s_1, s_4, s_3, s_1, s_3, s_2) \\
& = P^5(e_{s_2}, s_3, s_4, s_1, s_4, s_2) + r(s_3, s_1) - P^5(e_{s_2}, s_4, s_3, s_1, s_3, s_2) - r(s_4, s_1) \\
& = P^3(e_{s_2}, s_3, s_4, s_2) + r(s_4, s_1) + r(s_3, s_1) - P^3(e_{s_2}, s_4, s_3, s_2) - r(s_3, s_1) - r(s_4, s_1) \\
& = w(s_2, s_3, s_4),
\end{aligned}$$

which concludes the proof.  $\square$

**Lemma 3.7.** For every  $e_{s_1} \in M$ ,  $s_2, s_3 \in S$ ,

$$P^3(e_{s_1}, s_2, s_3, s_1) - e_{s_1} = \frac{1}{2}[w(s_1, s_2, s_3) + r(s_1, s_2) + r(s_2, s_3) + r(s_3, s_1)]$$

*Proof.* Note the identity which follows taking  $e'_{s_1} = P^3(e_{s_1}, s_3, s_2, s_1)$

$$\begin{aligned}
& P^3(e_{s_1}, s_2, s_3, s_1) - e_{s_1} = P^3(e'_{s_1}, s_2, s_3, s_1) - e'_{s_1} \\
& = P^3(P^3(e_{s_1}, s_3, s_2, s_1), s_2, s_3, s_1) - P^3(e_{s_1}, s_3, s_2, s_1) \\
& = [P^6(e_{s_1}, s_3, s_2, s_1, s_2, s_3, s_1) - e_{s_1}] + [e_{s_1} - P^3(e_{s_1}, s_3, s_2, s_1)] \\
& = \{[P(e_{s_1}, s_3, s_1) - e_{s_1}] + r(s_3, s_2) + r(s_2, s_1)\} + [e_{s_1} - P^3(e_{s_1}, s_3, s_2, s_1)] \\
& = r(s_1, s_3) + r(s_3, s_2) + r(s_2, s_1) + [e_{s_1} - P^3(e_{s_1}, s_3, s_2, s_1)],
\end{aligned}$$

thus

$$\begin{aligned}
& w(s_1, s_2, s_3) = [P^3(e_{s_1}, s_2, s_3, s_1) - e_{s_1}] + [e_{s_1} - P^3(e_{s_1}, s_3, s_2, s_1)] \\
& = 2[P^3(e_{s_1}, s_2, s_3, s_1) - e_{s_1}] - \{r(s_1, s_3) + r(s_3, s_2) + r(s_2, s_1)\}.
\end{aligned}$$

$\square$

**Definition 3.8.** Given a cyclic sequence of points, choose a point and denote it  $s_0$ , then, following the order of the sequence, denote the others  $s_1, s_2, \dots, s_k = s_0$ . The *flux*  $F(s_0 s_1 \dots s_{k-1})$  of the cyclic sequence is the quantity

$$F(s_0 s_1 \cdots s_{k-1}) = \frac{1}{2} \sum_{0 \leq i \leq k-1} w(s_0, s_i, s_{i+1}). \quad (10)$$

This definition in order to make sense must be independent of the chosen first element  $s_0$ , that is, it must be

$$F(s_0 s_1 \cdots s_{k-1}) = \frac{1}{2} \sum_{0 \leq i \leq k-1} w(s_j, s_i, s_{i+1}). \quad (11)$$

This is the case because using Eq. (9)

$$\frac{1}{2} \sum_{0 \leq i \leq k-1} w(s_j, s_i, s_{i+1}) - \frac{1}{2} \sum_{0 \leq i \leq k-1} w(s_0, s_i, s_{i+1}) \quad (12)$$

$$= \frac{1}{2} \sum_{0 \leq i \leq k-1} [w(s_j, s_i, s_{i+1}) - w(s_0, s_i, s_{i+1})] \quad (13)$$

$$= \frac{1}{2} \sum_{0 \leq i \leq k-1} [w(s_0, s_j, s_i) - w(s_0, s_j, s_{i+1})] = 0. \quad (14)$$

Thus to every closed oriented polygonal path in space there corresponds a quantity called flux. It is easy to check that if the orientation of the path is inverted the flux changes sign. Sometimes the flux will be called *holonomy*, see next section.

### 3.1 The radar distance and a bound for $w$

The quantity

$$d_r(s_0, s_1) = \frac{1}{2} r(s_0, s_1)$$

is also known as *radar distance*. Its interpretation as distance is obvious in special relativity and for an inertial reference frame, because in this particular case, using canonical Minkowski coordinates, it is easy to prove that it coincides with the usual Euclidean distance. However, as far as I know, no proof has ever been offered that  $d_r$  is a distance in more general situations, and in particular in presence of rotation. Note that in general relativity, even for a stationary frame with covariant velocity  $u^\alpha = k^\alpha / \sqrt{-k \cdot k}$ , this distance does not coincide with that calculated with the projected metric  $u_\alpha u_\beta + g_{\alpha\beta}$ , the reason being that the projection of the light beam selected with the Fermat's principle may depend on the direction considered, i.e. from  $s_0$  to  $s_1$ , or from  $s_1$  to  $s_0$ . In particular the distance so defined does not coincide with the length of a suitable geodesic on  $S$ .

**Theorem 3.9.** *The function  $d_r : S \times S \rightarrow [0, +\infty)$  (and hence  $r$ ) is a distance, that is*

(i) For every  $s_0, s_1 \in S$ ,  $d_r(s_0, s_1) \geq 0$  and the equality holds iff  $s_0 = s_1$ .

(ii) For every  $s_1, s_2, s_3 \in S$ ,  $d_r(s_1, s_3) \leq d_r(s_1, s_2) + d_r(s_2, s_3)$ .

*Proof.* Statement (i) follows trivially from property (c) of  $P$ . For statement (ii) note that from Fermat's condition on  $P$

$$P^2(e_{s_1}, s_2, s_3) - P(e_{s_1}, s_3) \geq 0,$$

applying  $P(\cdot, s_1)$

$$[P^3(e_{s_1}, s_2, s_3, s_1) - e_{s_1}] + [e_{s_1} - P^2(e_{s_1}, s_3, s_1)] \geq 0,$$

and from lemma 3.7

$$\frac{1}{2}w(s_1, s_2, s_3) + d_r(s_1, s_2) + d_r(s_2, s_3) - d_r(s_3, s_1) \geq 0. \quad (15)$$

Repeat the argument after the odd permutation  $(s_1, s_2, s_3) \rightarrow (s_3, s_2, s_1)$

$$\frac{1}{2}w(s_3, s_2, s_1) + d_r(s_3, s_2) + d_r(s_2, s_1) - d_r(s_1, s_3) \geq 0.$$

sum the two equations so obtained

$$d_r(s_3, s_2) + d_r(s_2, s_1) - d_r(s_1, s_3) \geq 0,$$

thus (ii) is proved.  $\square$

It is natural to introduce the *radar length*  $L_r$  of a polygonal path  $s_0 s_1 s_2 \dots s_k$  as

$$L_r(s_0 s_1 s_2 \dots s_k) = d_r(s_0, s_1) + d_r(s_1, s_2) + \dots + d_r(s_{k-1}, s_k). \quad (16)$$

**Theorem 3.10.** *The Sagnac function  $w(s_1, s_2, s_3)$  satisfies the bound*

$$|w(s_1, s_2, s_3)| \leq 2 \min\{d_r(s_1, s_2), d_r(s_2, s_3), d_r(s_3, s_1)\} \leq \frac{2}{3}L_r(s_1 s_2 s_3). \quad (17)$$

*Proof.* The proof goes as that of theorem 3.9 up to Eq. (15). Here consider the even permutation  $(s_1, s_2, s_3) \rightarrow (s_2, s_3, s_1)$  and repeat the argument which leads to Eq. (15) to obtain

$$\frac{1}{2}w(s_2, s_3, s_1) + d_r(s_2, s_3) + d_r(s_3, s_1) - d_r(s_1, s_2) \geq 0.$$

Summing this inequality with Eq. (15)

$$w(s_1, s_2, s_3) \geq -2d_r(s_2, s_3).$$

Consider now the inequality obtained from this one through the replacement  $(s_1, s_2, s_3) \rightarrow (s_1, s_3, s_2)$

$$w(s_1, s_3, s_2) \geq -2d_r(s_3, s_2) \Rightarrow w(s_1, s_2, s_3) \leq 2d_r(s_2, s_3).$$

and hence  $|w(s_1, s_2, s_3)| \leq 2d_r(s_2, s_3)$ . Rewriting this equation after the even permutations  $(s_1, s_2, s_3) \rightarrow (s_2, s_3, s_1) \rightarrow (s_3, s_1, s_2)$ , gives the thesis.  $\square$

It is well known that in general relativity the Sagnac effect over the path  $\sigma$  is given by the integral of the vorticity 2-form over a surface  $\Sigma$  such that  $\sigma = \partial\Sigma$  [7, 11] (this formula is obtained from Eq. (23) of [11]).

$$2 \int_{\Sigma} w_{ij} dx^i \wedge dx^j. \quad (18)$$

As a consequence, for small area elements the Sagnac effect is proportional to the area and to the scalar product of the vorticity vector with the normal to the area element. In other words, provided the area element is small, the Sagnac effect goes quadratically with the size (diameter) of the surface considered. The bound (17) proves that this quadratic behavior can not hold for large areas because the Sagnac effect is *linearly* bounded with respect to the size of the surface. This bound is satisfied for small areas because of the mentioned quadratic behavior. As the area increases the vorticity vector must (i) decrease in magnitude, (ii) have an increasing angle with respect to the surface normal (possibly with a change of sign of the scalar product as it happens on the equipotential surface of the earth).

**Theorem 3.11.** *Every round-trip time  $P^k(e_{s_0}, s_1, s_2, \dots, s_{k-1}, s_0) - e_{s_0}$  can be expressed as follows*

$$P^k(e_{s_0}, s_1, s_2, s_3 \dots, s_{k-1}, s_0) - e_{s_0} = F(s_0 s_1 \dots s_{k-1}) + L_r(s_0 s_1 \dots s_{k-1}). \quad (19)$$

In analogy with gauge theories the first term of the right-hand side can be called *holonomy* whereas the last term of the right-hand side can be called *dynamic phase* [11]. A consequence of this formula is that the Sagnac effect over a polygonal path equals twice the holonomy because the dynamic phase cancels out.

*Proof.* It is a consequence of lemma 3.7 together with lemma 3.2 and the definitions of flux and radar length. Triangulate the path  $s_0 \rightarrow s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow s_4 \rightarrow \dots$  as follows

$$s_0 \rightarrow s_1 \rightarrow s_2 \rightarrow s_0 \rightarrow s_2 \rightarrow s_3 \rightarrow s_0 \rightarrow s_3 \rightarrow s_4 \rightarrow s_0 \rightarrow s_4 \dots$$

taking into account that this operation adds a term  $r(s_0, s_2) + r(s_0, s_3) + r(s_0, s_4) + \dots$  to the round-trip time. The path so triangulated can be disjoined into the sum of the round trip times of the single triangles (they are triangle only in the sense that they are determined by the three vertices) which by lemma 3.7 can also be expressed as a sum of  $w$  and  $r$  terms

$$\begin{aligned} P^k(e_{s_0}, s_1, s_2, s_3 \dots, s_{k-1}, s_0) - e_{s_0} = & \frac{1}{2} [w(s_0, s_1, s_2) + w(s_0, s_2, s_3) + w(s_0, s_3, s_4) \\ & + \dots + w(s_0, s_{k-2}, s_{k-1}) + r(s_0, s_1) + r(s_1, s_2) + r(s_2, s_3) + r(s_3, s_4) \\ & + \dots + r(s_{k-1}, s_0)]. \end{aligned}$$

□

A consequence of the last theorem is that every observable of the theory is a functional of functions  $w$  and  $r$ , unless additional structure is introduced.

## 4 Einstein's synchronization

Given a choice of event  $e_s \in \mathbb{E}_s$ , any other event  $e \in \mathbb{E}_s$  on the same fiber can be identified with a real number  $t(e) = e - e_s$  where zero corresponds to  $e_s$ . A section is a map  $\sigma : S \rightarrow M$  such that  $\pi \circ \sigma = Id_S$ . It sends  $s \rightarrow e_s$ . Thus given a section and  $e \in M$  one has a space component  $s = \pi(s)$  and a time coordinate  $t(e) = e - e_s$ . The problem of synchronization theory is the selection of a section, or equivalently, of a zero level at each fiber. Clearly, given a method of synchronization that works there always remains an overall translational invariance so that the zero level at least for a given fiber can be chosen arbitrarily.

The usual method is Einstein's. If  $e_{s_1}$  is the stipulated zero level of  $s_1$ 's fiber then the zero level of  $s_2$ 's fiber is, according to Einstein,

$$e_{s_2} = P(e_{s_1}, s_2) - \frac{r(s_1, s_2)}{2}. \quad (20)$$

The Einstein's synchronization convention would be satisfactory if it could be proved to be reflective, symmetric and transitive. As for reflectivity simply replace  $s_2$  with  $s_1$  on the right-hand side to find the identity  $e_{s_2} = e_{s_1}$ . Symmetry follows with a little algebra using the symmetry of  $r$

$$\begin{aligned} & P(e_{s_2}, s_1) - \frac{r(s_2, s_1)}{2} \\ &= P(P(e_{s_1}, s_2) - \frac{r(s_1, s_2)}{2}, s_1) - \frac{r(s_2, s_1)}{2} \\ &= P^2(e_{s_1}, s_2, s_1) - r(s_1, s_2) = e_{s_1}. \end{aligned}$$

Note the usefulness of the introduced mathematical structure. It has reduced the verification of these properties into a matter of algebra. There is no need to bother oneself with a description of the propagation of the signals.

Unfortunately in general Einstein's synchronization is not transitive. As a matter of fact, the relevance of the property  $\mathbf{z} = \mathbf{0}$  for its very definition to make sense was not immediately recognized (if  $\mathbf{z} = \mathbf{0}$  does not hold then two clocks to which Einstein's method has been applied may not be found synchronized at a later time). The fact that the symmetry follows from  $\mathbf{z} = \mathbf{0}$  was pointed out by L. Silberstein [24] in 1914. He also suggested that given the property  $\mathbf{z} = \mathbf{0}$  the transitivity of Einstein's synchronization method is equivalent to the so called Reichenbach round-trip condition which states that the signal covering a triangle lasts a time which is independent of the direction followed around the triangle. In our notation

$$\Delta: \text{For every } e_{s_0} \in M, s_1, s_2 \in S, P^3(e_{s_0}, s_1, s_2, s_0) = P^3(e_{s_0}, s_2, s_1, s_0)$$

Because of theorem 3.11 it amounts to the requirement  $w = F = 0$ .

A proof was given by H. Reichenbach who, however, missed to realize the need and importance of the tacit assumption  $\mathbf{z} = \mathbf{0}$ . H. Weyl [25] gave a similar

proof based on a stronger assumption known as Weyl's round-trip condition, which states that the time it takes light to cover a closed polygonal path of length  $L$  is  $L$  (in suitable units). Weyl missed the relevance of assumption  $z = \mathbf{0}$  too (see the discussion in [28]). Weyl's condition makes sense only if a distance is defined over  $S$ , thus in some sense it is less general than  $\Delta$ . However, the following result holds

**Theorem 4.1.** *In a synchronization structure Weyl's round-trip condition is equivalent to Reichenbach's provided the distance used in Weyl's condition is the radar distance.*

*Proof.* It is trivial because Weyl's condition reads

$$P^k(e_{s_0}, s_1, s_2, s_3 \dots, s_{k-1}, s_0) - e_{s_0} = L_r(s_0, s_1, s_2, s_3 \dots, s_{k-1}),$$

while Reichenbach's condition reads  $F(s_0, s_1, s_2, s_3 \dots, s_{k-1}) = 0$ , and they are equivalent because of theorem 3.11.  $\square$

The first clear proof of the equivalence between the transitivity of Einstein's synchronization and  $\Delta$  (provided  $z = \mathbf{0}$  holds) was given by A. Macdonald [26]. The proof is not repeated here because it will be obtained in the next section as a particular case of the transitivity proof for a more general synchronization method.

## 5 The new synchronization method

Assume there is a natural way of writing function  $w$  as a 2-coboundary

$$w(s_1, s_2, s_3) = \delta(s_1, s_2) + \delta(s_2, s_3) + \delta(s_3, s_1), \quad (21)$$

where  $\delta : S \times S \rightarrow \mathbb{R}$  is a skew-symmetric function. The generalized synchronization which replaces Einstein's (Eq. (20)) is given by the formula

$$e_{s_2} = P(e_{s_1}, s_2) - \frac{r(s_1, s_2) + \delta(s_1, s_2)}{2}. \quad (22)$$

**Theorem 5.1.** *Let  $\delta : S \times S \rightarrow \mathbb{R}$  be a skew-symmetric function which satisfies Eq. (21). The synchronization method given by Eq. (22) is reflexive, symmetric and transitive, thus being an equivalence relation it leads to a foliation of  $M$ .*

*Conversely, for every foliation represented by a section  $s \rightarrow e_s$  there is a skew-symmetric function  $\delta$ , defined by Eq. (22), which satisfies Eq. (21) and leads to that foliation.*

*Proof.* It is reflexive because if  $s_2 = s_1$ , it gives  $e_{s_2} = e_{s_1}$ . It is symmetric indeed

$$\begin{aligned} P(e_{s_2}, s_1) - \frac{r(s_2, s_1) + \delta(s_2, s_1)}{2} &= P(P(e_{s_1}, s_2) - \frac{r(s_1, s_2) + \delta(s_1, s_2)}{2}, s_1) \\ &\quad - \frac{r(s_2, s_1) + \delta(s_2, s_1)}{2} \\ &= P^2(e_{s_1}, s_2, s_1) - r(s_1, s_2) = e_{s_1}. \end{aligned}$$

Finally, it is transitive indeed assume that  $s_1$  and  $s_2$  are synchronized and that  $s_2$  and  $s_3$  are synchronized

$$e_{s_2} = P(e_{s_1}, s_2) - \frac{r(s_1, s_2) + \delta(s_1, s_2)}{2}, \quad (23)$$

$$e_{s_3} = P(e_{s_2}, s_3) - \frac{r(s_2, s_3) + \delta(s_2, s_3)}{2}, \quad (24)$$

where  $e_{s_1}$ ,  $e_{s_2}$  and  $e_{s_3}$  give the zero level at the corresponding fibers according to the above synchronization method. From Eqs. (23) and (24)

$$\begin{aligned} & P^3(e_{s_1}, s_2, s_3, s_1) - e_{s_1} \\ &= P^2(e_{s_2}, s_3, s_1) - e_{s_1} + \frac{r(s_1, s_2) + \delta(s_1, s_2)}{2} \\ &= P(e_{s_3}, s_1) - e_{s_1} + \frac{r(s_2, s_3) + \delta(s_2, s_3)}{2} + \frac{r(s_1, s_2) + \delta(s_1, s_2)}{2} \end{aligned}$$

Recalling lemma 3.7 and Eq. (21) it follows

$$e_{s_1} = P(e_{s_3}, s_1) - \frac{r(s_3, s_1) + \delta(s_3, s_1)}{2} \quad (25)$$

which states that  $s_1$  and  $s_3$  are synchronized.

For the converse, given the section  $s \rightarrow e_s$  and defined  $\delta : S \times S \rightarrow \mathbb{R}$  as

$$\delta(s_1, s_2) = 2[P(e_{s_1}, s_2) - e_{s_2}] - r(s_1, s_2),$$

function  $\delta$  is skew-symmetric, indeed

$$\begin{aligned} \delta(s_2, s_1) &= 2[P(e_{s_2}, s_1) - e_{s_1}] - r(s_2, s_1) \\ &= 2\{P([P(e_{s_1}, s_2) - \frac{r(s_1, s_2) + \delta(s_1, s_2)}{2}], s_1) - e_{s_1}\} - r(s_2, s_1) \\ &= 2\{P(P(e_{s_1}, s_2), s_1) - e_{s_1}\} - r(s_1, s_2) - \delta(s_1, s_2) - r(s_2, s_1) \\ &= -\delta(s_1, s_2). \end{aligned}$$

It remains to prove that  $\delta$  satisfies Eq. (21)

$$\begin{aligned} \delta(s_1, s_2) + \delta(s_2, s_3) + \delta(s_3, s_1) &= 2\{[P(e_{s_1}, s_2) - e_{s_2}] + [P(e_{s_2}, s_3) - e_{s_3}] \\ &\quad + [P(e_{s_3}, s_1) - e_{s_1}]\} - [r(s_1, s_2) + r(s_2, s_3) + r(s_3, s_1)] \end{aligned}$$

now, use the identities

$$\begin{aligned} P(e_{s_1}, s_2) - e_{s_2} &= P^2(e_{s_1}, s_2, s_3) - P(e_{s_2}, s_3) \\ P^2(e_{s_1}, s_2, s_3) - e_{s_3} &= P^3(e_{s_1}, s_2, s_3, s_1) - P(e_{s_3}, s_1) \end{aligned}$$

to obtain

$$\begin{aligned} \delta(s_1, s_2) + \delta(s_2, s_3) + \delta(s_3, s_1) &= 2\{P^3(e_{s_1}, s_2, s_3, s_1) - e_{s_1}\} \\ &\quad - [r(s_1, s_2) + r(s_2, s_3) + r(s_3, s_1)] = w(s_1, s_2, s_3) \end{aligned}$$

where in the last step lemma 3.7 has been used. □



*Remark 5.2.* Physically Eq. (22) states that in order to synchronize clock  $s_2$  with clock  $s_1$  one has to send a signal from  $s_1$  to  $s_2$  along with the information of the time  $t_1$  measured by  $s_1$  at the instant of the signal departure. At the instant of arrival clock  $s_2$  is set so that it measures a time  $t_2 = t_1 + \frac{r(s_1, s_2) + \delta(s_1, s_2)}{2}$  where  $r(s_1, s_2)$  and  $\delta(s_1, s_2)$  must be determined in advance. In short there is a correction  $\delta(s_1, s_2)/2$  with respect to Einstein's method.

The previous theorem does not state that a section  $s \rightarrow e_s$  exist, or equivalently, it does not state that a skew-symmetric function which satisfies Eq. (21) exists.

Also the existence of a function  $\delta$  such that Eq. (21) holds by itself does not solve the problem of synchronization. Indeed, the function  $\delta$  must be an *observable* otherwise the synchronization method described here would not have any practical value. Another condition to be imposed on  $\delta$  is that it must vanish whenever  $w$  vanishes so that the usual Einstein's synchronization is recovered in this case.

The problem of the existence and observability of function  $\delta$  is answered by the following

**Theorem 5.3.** *Let  $\mu$  be a normalized measure on (a suitable  $\sigma$ -algebra of)  $S$ ,  $\int_S d\mu(s) = 1$ , then*

$$\delta(s_1, s_2) = \int_S w(s_1, s_2, s) d\mu(s), \quad (26)$$

*satisfies Eq. (21),  $\int_S \delta(s, s') d\mu(s') = 0$ , and vanishes if  $w = 0$ . Conversely, given  $\delta : S \times S \rightarrow \mathbb{R}$  skew-symmetric, such that  $\int_S \delta(s, s') d\mu(s') = 0$ , defined  $w$  through Eq. (21) it follows Eq. (26).*

*Proof.* It suffices to make use of the 2-cocycle condition, Eq. (9),

$$\begin{aligned} \delta(s_1, s_2) + \delta(s_2, s_3) + \delta(s_3, s_1) &= \int_S [w(s_1, s_2, s) + w(s_2, s_3, s) + w(s_3, s_1, s)] d\mu(s) \\ &= \int_S w(s_1, s_2, s_3) d\mu(s) = w(s_1, s_2, s_3). \end{aligned}$$

The other statements are trivial.  $\square$

*Remark 5.4.* The previous theorem does not state that every function  $\delta'$  which satisfies Eq. (21) and vanishes whenever  $w = 0$ , is given by Eq. (26). Assume there is another skew-symmetric function  $\delta' : S \times S \rightarrow \mathbb{R}$  which satisfies Eq. (21), then defined  $\Delta = \delta' - \delta$  it is (1-cocycle condition)

$$\Delta(s_1, s_2) + \Delta(s_2, s_3) + \Delta(s_3, s_1) = 0.$$

Define  $\eta : S \rightarrow \mathbb{R}$ , with  $\eta(s_1) = \int_S \Delta(s_1, s) d\mu(s)$ , then

$$\delta'(s_1, s_2) = \delta(s_1, s_2) + \eta(s_1) - \eta(s_2), \quad (27)$$

indeed

$$\begin{aligned}
\delta(s_1, s_2) + \eta(s_1) - \eta(s_2) &= \delta'(s_1, s_2) - \int_S \Delta(s_1, s_2) d\mu(s) \\
&\quad + \int_S \Delta(s_1, s) d\mu(s) - \int_S \Delta(s_2, s) d\mu(s) \\
&= \delta'(s_1, s_2) - \int_S [\Delta(s_1, s_2) + \Delta(s_2, s) + \Delta(s, s_1)] d\mu(s) = \delta'(s_1, s_2).
\end{aligned}$$

Thus  $\delta'$  differs from  $\delta$  by a 1-coboundary term which vanishes if  $w = 0$ .

Although we gave no proof that  $\delta$  must necessarily be given by the expression (26), it is clear that the simplest choice for  $\delta$  is given by that equation. Thus the alternatives to the Poincaré-Einstein's synchronization convention will pass through the selection of a measure on  $S$ .

The synchronization structure does not provide a measure, but depending on the problem considered, a natural measure on  $S$  can be given.

For a network of computers each point of  $S$  represents a computer's clock and as measure  $\mu$  one can take the discrete measure that assign the same relevance to every node. Different choices can be also considered depending on the importance of the computer in the network.

As for general relativity, here  $S$  is the quotient manifold generated by a congruence of timelike curves. Let  $u$ ,  $u^\mu u_\mu = -1$ , be the normalized vector field which generates the congruence, and assume that  $u^\mu = k^\mu / \sqrt{-k^\alpha k_\alpha}$  where  $k$  is a timelike Killing vector field. The tensor  $\varepsilon_{\alpha\beta\gamma} = u^\mu \epsilon_{\mu\alpha\beta\gamma}$ , where  $\epsilon_{\mu\alpha\beta\gamma}$  is the volume form on  $M$ , projects into a volume form on  $S$ , i.e. the volume form of the quotient metric represented on  $M$  by  $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$  (see [27]). Thus it is natural to choose  $\mu$  coincident up to a constant factor with this volume form as it depends only on the congruence and hence on the definition of frame. Note, however, that the quotient volume form must have a finite integral over  $S$  otherwise the proportionality constant can not be chosen so as to normalize  $\mu$ . Note also that  $z = \mathbf{0}$  is satisfied on the equipotential slices, that is on those sets for which  $k^\alpha k_\alpha = \text{const}$ .

In order to satisfy  $z = \mathbf{0}$  over  $S$  even when  $k^\alpha k_\alpha$  is not constant everywhere, one can replace the metric  $g$  with the conformal metric  $g/(-k \cdot k)$ , and the space metric with the optical metric  $h/(-k \cdot k)$ . In this way  $k$  is sent into a timelike Killing field of constant norm. The new synchronization procedure can be applied safely and the theoretical foliation obtained for the conformal spacetime can be finally passed to the original spacetime.

One of most important applications is in the problem of synchronization around a planet, say, the earth. If the spacetime of the planet is described by a stationary metric where the planet congruence is generated by the Killing vector then it is convenient to slice the quotient  $Q$  (for this application the quotient of the congruence is denoted  $Q$ , the set  $S$  is defined below) into equipotential slices (the redshift between two points on the same slice vanishes). Then chosen an equipotential slice  $S$  (say the surface of the earth) there is a natural area form induced by  $h_{\mu\nu}$ . This area form can be normalized to obtain  $\mu$ . Thus the

new synchronization algorithm can be applied to lead to a natural foliation of the spacetime.

It is quite easy to show that in the Schwarzschild spacetime,

$$g = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

if  $S$  is a surface  $r = \text{const.}$  on the quotient space  $Q$  of coordinates  $(r, \theta, \varphi)$  (here  $k = \partial_t$ ), then the synchronization method gives a foliation that coincides with the usual coordinate  $t$  (because  $w$  vanishes identically). Similar considerations for the Kerr spacetime seem much more complex, in the first place because lightlike geodesic propagating from space point  $s_1$  to  $s_2$  or from  $s_2$  to  $s_1$  may have different projections on the quotient  $Q$ .

The determination of the coordinate time associated to our synchronization convention for various interesting metrics deserves to be investigated and will require further work.

## 6 Conclusions

A minimal mathematical structure has been introduced to study the problem of synchronization in different contexts. Two observables have been introduced, the function  $r$  giving the two-way delay and  $w$  giving the Sagnac effect over a ‘triangular’ path. The Poincaré-Einstein’s method is transitive only if  $w$  vanishes and there is no redshift (property  $z = 0$  holds). A new method has been introduced which reduces to Poincaré-Einstein’s if  $w = 0$  but which is transitive even for  $w \neq 0$ . The new method depends on a normalized measure  $\mu$  on the space  $S$ , which depends on the problem considered and which is selected according to simplicity criteria. As an example the problem of the synchronization of clocks at the equipotential surface of a planet can be solved using the new method. In practice (remark 5.2) it consists in a correction to the usual Poincaré-Einstein’s method of synchronization, the correction being obtained through a suitable integral of the Sagnac effect over  $S$  (see Eq. (26)).

It must be said that although the non-transitivity of the Poincaré-Einstein’s method has been known for a long time almost no publication has ever appeared which proposed a correction to that method in order to accomplish transitivity (to the best of my knowledge the only published attempt is due to the author who presented an approximate local approach in [19]). This lack of contributions seems more related to the somewhat widespread opinion that this goal was difficult to achieve rather than on a lack of interest for the problem. In this sense the solution proposed in this work might have particular value.

The exact calculation of the integral (5.3) given the spacetime metric may be difficult but in practice it can be approximated with a sum over a suitable lattice of clocks over  $S$ . Thus the method has practical value although it is not meant as a replacement for the GPS “common view” method. The GPS synchronization has an accuracy which at present cannot be reached with the new method because of the servers’ instabilities (recall that the fact that the

signal is ‘slow’ on the cables or the computers with respect to a suitable external time plays no role, see remark 2.4), that is, because the condition  $z = \mathbf{0}$  is satisfied only approximatively. However, the issue as to whether the new method could become competitive is worth studying.

Perhaps the most significant consequence is that, contrary to what could be expected, *there is*, in many cases, a natural splitting of spacetime into space and time and that this result is exact (provided the assumptions are satisfied). This surprising fact may prove to be useful in quantum gravity, where the lack of such a privileged splitting has come to be known as “the problem of time”.

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